RATIONAL POINTS ON CERTAIN ELLIPTIC SURFACES

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Abstract. Let $\mathcal{E}_f : y^2 = x^3 + f(t)x$, where $f \in \mathbb{Q}[t] \setminus \mathbb{Q}$, and let us assume that $\deg f \leq 4$. In this paper we prove that if $\deg f \leq 3$, then there exists a rational base change $t \mapsto \varphi(t)$ such that there is a non-torsion section on the surface $\mathcal{E}_{f\circ\varphi}$. A similar theorem is valid in case when $\deg f = 4$ and there exists $t_0 \in \mathbb{Q}$ such that infinitely many rational points lie on the curve $E_{t_0} : y^2 = x^3 + f(t_0)x$. In particular, we prove that if $\deg f = 4$ and f is not an even polynomial, then there is a rational point on $\mathcal{E}_f$. Next, we consider a surface $\mathcal{E}^g : y^2 = x^3 + g(t)$, where $g \in \mathbb{Q}[t]$ is a monic polynomial of degree six. We prove that if the polynomial g is not even, there is a rational base change $t \mapsto \psi(t)$ such that on the surface $\mathcal{E}^{g\circ\psi}$ there is a non-torsion section. Furthermore, if there exists $t_0 \in \mathbb{Q}$ such that on the curve $E^{t_0} : y^2 = x^3 + g(t_0)$ there are infinitely many rational points, then the set of these t_0 is infinite. We also present some results concerning diophantine equation of the form $x^2 - y^3 - g(z) = t$, where t is a variable.

1. Introduction

Let $\mathcal{E}$ be an elliptic surface given by the equation

$$\mathcal{E} : y^2z = x^3 + A(t)xz^2 + B(t)z^3,$$

where $A,\ B \in \mathbb{Q}[t]$. The discriminant for $\mathcal{E}$ is defined by $\Delta(t) = -16(4A(t)^3 + 27B(t)^2)$, while j-invariant is defined by $j(t) = -1728(4A(t))^3/\Delta(t)$. We call the surface $\mathcal{E}$ isotrivial if its j-invariant is constant. We say that the surface $\mathcal{E}$ splits if there exists an elliptic curve C such that $\mathcal{E} \simeq C \times \mathbb{P}$ over $\mathbb{Q}$. In the sequel by an elliptic surface we mean a non-split one. There is a natural projection on $\mathcal{E}$ given by $\pi : \mathcal{E} \ni ([x : y : z],\ t) \mapsto t \in \mathbb{P}$. The mapping $\sigma : \mathbb{P} \to \mathcal{E}$ fulfilling a condition $\pi \circ \sigma = id_{\mathbb{P}}$ will be called a section on $\mathcal{E}$. Throughout the paper, by a section we mean one defined over $\mathbb{Q}$. Let us note that there is always zero section on $\mathcal{E}$ given by $\sigma_0 = ([0 : 1 : 0],\ t)$. We can look on the surface $\mathcal{E}$ as on an elliptic curve defined over $\mathbb{Q}(t)$. Therefore, we have the Mordell-Weil type theorem for $\mathcal{E}$, which says that the set of sections (or equivalently points on $\mathcal{E}$ defined over $\mathbb{Q}(t)$) forms a finitely generated abelian group. Because for all but finitely many $t \in \mathbb{Q}$ a fibre $\mathcal{E}$ of the mapping π is an elliptic curve, a natural question arises: what can we say about the set of $t \in \mathbb{Q}$ such that the elliptic curve $\mathcal{E}_t$ has a positive rank? In case when we have a non-torsion section on $\mathcal{E}$, this question follows trivially from Silverman's specialization theorem ([8], page 368). This theorem says that for all but finitely many $t \in \mathbb{Q}$ the curve $\mathcal{E}_t$ has a positive rank. The second interesting question concerns the existence of rational curves on the surface $\mathcal{E}$. Let us note that the existence of a rational curve on $\mathcal{E}$, say $(x(u),\ y(u),\ \psi(u))$, gives us a

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rational base change $t = \psi(u)$ such that $\sigma = (x(u),\ y(u))$ is a section on the surface $\mathcal{E}_\psi : y^2 = x^3 + A(\psi(u))x + B(\psi(u))$. As we will see, in many cases σ is a non-torsion section. It is worth noting here that the problem of this kind was considered in Whitehead's paper ([9]). He proved that a rational curve lies on the surface given by the equation $z^2 = f(x, y)$, where $f \in \mathbb{Q}[x, y]$ and $\deg f = 3$. It is easy to see that the surface defined in this way is birationally equivalent to the surface $\mathcal{E}$ for certain $A,\ B \in \mathbb{Q}[t]$ with $\deg A \leq 2,\ \deg B \leq 3$.

Let us also note that the existence of a rational base change $t = \psi(u)$ such that we have a non-torsion section on $\mathcal{E}_\psi$, and Silverman's specialization theorem imply that for all but finitely many $u \in \mathbb{Q}$, each fibre $\mathcal{E}_{\psi(u)}$ has a dense set of rational points.

In Section 2 we consider a surface of the form $\mathcal{E}_f : y^2 = x^3 + f(t)x$, where $f \in \mathbb{Q}[t]$ and $\deg f \leq 4$. If $\deg f \leq 3$, then we show that there exists a rational base change $t = \varphi(s)$ such that there is a non-torsion section on the surface $\mathcal{E}_{f\circ\varphi}$. A similar theorem is proved in case when $\deg f = 4$ and with the assumption that there exists $t_0 \in \mathbb{Q}$ such that there are infinitely many rational points on the curve $E_{t_0} : y^2 = x^3 + f(t_0)x$. In particular, we prove that if the polynomial of degree four is not even, then there is a non-trivial rational point on the surface $\mathcal{E}_f$.

In Section 3 we consider a surface of the form $\mathcal{E}^g : y^2 = x^3 + g(t)$, where $g \in \mathbb{Q}[t]$ is a monic polynomial of degree six. In this case we prove that if g is not an even polynomial, then there is a rational base change $t = \chi(u)$ such that there exists a non-torsion section on $\mathcal{E}^{g\circ\chi}$. Moreover, in case when the polynomial g is even, and there exists $t_0 \in \mathbb{Q}$ such that the curve $E^{t_0} : y^2 = x^3 + g(t_0)$ contains infinitely many rational points, then the set of $t_0 \in \mathbb{Q}$ such that E^{t_0} has a positive rank is infinite.

In Section 4 we present some results concerning diophantine equation of the form

$$x^2 - y^3 - g(z) = t,$$

where $g(z) = z^6 + az^4 + bz^3 + cz^2 + dz + e \in \mathbb{Z}[z]$ and t is a variable. We will deal with the solution of this equation in the ring of polynomials $\mathbb{Q}[t]$. In particular, we prove that if $a \equiv 1 \pmod 2$ and $b \neq 0$, then the above equation has infinitely many solutions in $\mathbb{Q}[t]$.

In Section 5 we present some results about rational points on certain non-isotrivial elliptic surfaces.

2. Rational points on $\mathcal{E}_f : y^2 = x^3 + f(t)x$

Let $f \in \mathbb{Q}[t] \setminus \mathbb{Q}$ and let us assume that $\deg f \leq 4$ and f has at least two different complex roots. For such f we consider a surface $\mathcal{E}_f$ given by the equation

$$\mathcal{E}_f : y^2 = x^3 + f(t)x.$$

Because f does not have a root of multiplicity four, our elliptic surface $\mathcal{E}_f$ is non-split. For a given $t \in \mathbb{Q}$ let us denote the curve $y^2 = x^3 + f(t)x$ by E_t. It is worth noting that for a fixed $t \in \mathbb{Q}$, the torsion part of the group $E_t(\mathbb{Q})$ is isomorphic to one of the following ([8], page 323): $\mathbb{Z}/4\mathbb{Z}$, if $f(t) = 4$; $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, if $-f(t)$ is a square; $\mathbb{Z}/2\mathbb{Z}$ if $f(t)$ does not fulfill any of these conditions. As an immediate consequence we obtain that if there is a rational base change $t = \beta(u)$ such that we have a section $\sigma = (x,\ y)$ with $y \neq 0$ on the surface $\mathcal{E}_{f\circ\beta}$, then σ is a non-torsion section.

We show the following

Theorem 2.1.

(1) *If* $\deg f \le 3$, *then there exists a rational base change* $t = \varphi(s)$ *such that there is a non-torsion section on the surface* $\mathcal{E}_{f\circ\varphi}$.

(2) *If* $\deg f = 4$ *and there is* $t_0 \in \mathbb{Q}$ *such that the curve* E_{t_0} *has infinitely many rational points, then there exists a rational base change* $t = \psi(r)$ *such that there is a non-torsion section on the surface* $\mathcal{E}_{f\circ\psi}$.

Proof. For the proof of our theorem it will be convenient to work with the surface $\mathcal{E}'_f$ given by the equation

$$\mathcal{E}'_f : XY^2 = X^2 + f(t),$$

which is birationally equivalent to $\mathcal{E}_f$ by the mapping $(x, y, t) = (X, XY, t)$ with inverse $(X, Y, t) = (x, y/x, t)$. Let us denote $F(X,\ Y,\ t) := XY^2 - X^2 - f(t)$.

Proof of part (1).

Let $f \in \mathbb{Q}[t]$ and $\deg f \le 3$. Without loss of generality we can assume that $f(t) = at^3 + bt^2 + ct + d$ for some $a,\ b,\ c,\ d \in \mathbb{Z}$ with $a \ne 0$ or $b \ne 0$. If $a = b = 0$, then f has degree 1 and if we put $t = (s^4 - d)/c$, the surface splits over $\mathbb{Q}(s)$. Let us put $X = pT + q,\ Y = rT + s,\ t = T$. For $X,\ Y,\ t$ defined in this way we obtain

$$F(X,\ Y,\ t) = a_0 + a_1T + a_2T^2 + a_3T^3,$$

where

$$a_0 = -d - q^2 + qs^2, \quad a_1 = -c - 2pq + 2qrs + ps^2,$$
$$a_2 = -b - p^2 + qr^2 + 2prs, \quad a_3 = -a + pr^2.$$

Let us note that the system of equations $a_2 = a_3 = 0$ has exactly one solution given by

$$p = \frac{a}{r^2}, \quad q = \frac{a^2 + br^4 - 2ar^3s}{r^6}. \tag{2.1}$$

For $p,\ q$ given in this way the equation $F(pT + q,\ rT + s,\ T) = 0$ has the root $T = -\varphi_1(r,\ s)/\varphi_2(r,\ s)$, where

$$\varphi_1(r,\ s) = a^4 + 2a^2br^4 + b^2r^8 + dr^{12} - 4ar^3(a^2s + br^4)s + r^6(3a^2 - br^4)s^2 + 2ar^9s^3,$$
$$\varphi_2(r,\ s) = r^4(2a^3 + 2abr^4 + cr^8 - 2r^3(3a^2 + br^4)s + 3ar^6s^2).$$

We have obtained a two-parametric solution of the equation defining the surface $\mathcal{E}'_f$. Let us define $\varphi(s) := -\varphi_1(1,\ s)/\varphi_2(1,\ s)$ and put $t = \varphi(s)$. We can see that we have a section $\sigma = (p\varphi(s) + q,\ (p\varphi(s) + q)(\varphi(s) + s))$ on the surface $\mathcal{E}_{f\circ\varphi}$. Because $(p\varphi(s) + q)(\varphi(s) + s)$ is a nonzero rational function, the section σ is not of order two, which proves that it is a non-torsion section.

Proof of part (2).

Because $\deg f = 4$, we can assume without loss of generality that $f(t) = at^4 + bt^2 + ct + d$ for certain $a,\ b,\ c,\ d \in \mathbb{Z}$ with $a \ne 0$. From the assumption, there exists $t_0 \in \mathbb{Q}$ such that $(x_0,\ y_0,\ t_0)$ is a rational point on $\mathcal{E}_f$ and $x_0 \ne 0$. Then the point $(x_0,\ y_0/x_0,\ t_0)$ is a rational point on the surface $\mathcal{E}'_f$.

Let us now put $X = pT^2 + qT + x_0,\ Y = rT + y_0/x_0,\ t = T + t_0$. For $X,\ Y,\ t$ defined in this way we get

$$F(X,\ Y,\ t) = (a_1T + a_2T^2 + a_3T^3 + a_4T^4)/x_0^2,$$

where

$$\begin{aligned}
a_1 &= x_0^2(c + 2bt_0 + 4at_0^3 - 2ry_0) + q(2x_0^3 - y_0^2),\\
a_2 &= x_0(bx_0 + q^2x_0 + 6at_0^2x_0 - r^2x_0^2 - 2qry_0) + p(2x_0^3 - y_0^2),\\
a_3 &= x_0(2pqx_0 - qr^2x_0 + 4at_0x_0 - 2pry_0),\\
a_4 &= (a + p^2 - pr^2)x_0^2.
\end{aligned}$$

If now $2x_0^3 - y_0^2 \neq 0$, then the system of equations $a_1 = a_2 = 0$ is triangular with respect to $p,\ q$. Because the curve E_{t_0} has infinitely many rational points, by fixed $a,\ b,\ c,\ d$ we will choose $x_0,\ y_0$ such that $2x_0^3 - y_0^2 \neq 0$, and the system $a_1 = a_2 = 0$ has a solution fulfilling the condition $p \neq 0$ or $q \neq 0$. Therefore, we obtain

(2.2)
$$q = -\frac{x_0^2(c + 2bt_0 + 4at_0^3 - 2ry_0)}{2x_0^3 - y_0^2},\ p = -\frac{x_0(bx_0 + q^2x_0 + 6at_0^2x_0 - r^2x_0^2 - 2qry_0)}{2x_0^3 - y_0^2}.$$

For $p,\ q$ defined in this way, the equation $F(pT^2+qT+x_0,\ rT+y_0/x_0,\ T+t_0) = 0$ has the triple root $T = 0$ and the root

$$T = -\frac{2pqx_0 - qr^2x_0 + 4at_0x_0 - 2pry_0}{(a + p^2 - pr^2)x_0} =: \psi(r) - t_0. \tag{2.3}$$

If we now put $t = \psi(r)$, then on the surface $\mathcal{E}_{f\circ\psi}$ there is a section $\sigma = (pT^2 + qT + x_0,\ (rT + y_0/x_0)(pT^2 + qT + x_0),\ r)$, where $p,\ q$ are given by (2.2) and T is given by (2.3). Since $(rT + y_0/x_0)(pT^2 + qT + x_0) \neq 0$, the section σ is not of order two, which proves that it is a non-torsion section. □

Here a natural and nontrivial question arises concerning the construction of polynomials f of degree four for which there is a rational point with $y \neq 0$ on the surface $\mathcal{E}_f$. It turns out that there exists a wide class of polynomials with this property.

Now we will show the following

Theorem 2.2. *If $f \in \mathbb{Q}[t]$, $\deg f = 4$ and $f(t) \neq f(-t)$, then there exists a rational base change $t = \varphi(u)$ such that on the surface $\mathcal{E}_{f\circ\varphi}$ we have a non-torsion section.*

Proof. Without loss of generality we can assume that $f(t) = at^4 + bt^2 + ct + d$ for certain $a,\ b,\ c,\ d \in \mathbb{Z}$, where $ac \neq 0$. Let u be a variable and let us put $x = au^2$ and treat our surface as a curve of degree 4 defined over $\mathbb{Q}(u)$, i.e. we consider the curve

$$C_1:\quad y^2 = a^2u^2t^4 + abu^2t^2 + acu^2t + adu^2 + a^3u^6 =: h_1(t).$$

Let us note that the point at infinity on the curve C_1 is rational. Let us now put $t = T,\ y = auT^2 + pT + q$. Then

$$(auT^2 + pT + q)^2 - h_1(T) = a_0 + a_1T + a_2T^2 + a_3T^3,$$

where

$$\begin{aligned}
a_0 &= -q^2 + adu^2 + a^3u^6, \quad a_1 = -2pq + acu^2,\\
a_2 &= -p^2 - 2aqu + abu^2, \quad a_3 = -2apu.
\end{aligned}$$

The system of equations $a_2 = a_3 = 0$ has a solution given by $p = 0,\ q = bu/2$. For $p,\ q$ defined in this way, the equation $(auT^2 + pT + q)^2 - h_1(T) = 0$ has the root

$$T = -\frac{-b^2 + 4ad + 4a^3u^4}{4ac} =: \varphi(u).$$

We have shown that with the assumption $ac \neq 0$ on the surface $\mathcal{E}_{f\circ\varphi}$ there is a section

$$\sigma_1 = \Big(au^2,\ \frac{(-b^4 - 8abc^2 + 8ab^2d - 16a^2d^2)u + 8a^3(b^2 - 4ad)u^5 - 16a^6u^9}{16ac^2}\Big),$$

which is clearly non-torsion. $\square$

From the above theorem we obtain two interesting corollaries

Corollary 2.3. *If $f(t) = at^4 + bt^2 + ct + d \in \mathbb{Z}[t]$, $a,\ c \in \{-1,\ 1\}$ and $b \equiv 0 \pmod 2$, then the diophantine equation $y^2 = x^3 + f(t)x$ has infinitely many solutions in integers.*

Corollary 2.4. *If $f \in \mathbb{Q}[t]$, $\deg f = 4$, f is not an even polynomial and f has at least two complex roots, then the diophantine equation $v^2 = u^4 + f(w)$ has infinitely many rational parametric solutions.*

Proof. Let us denote $S : v^2 = u^4 + f(w)$. Using the method described in ([6], page 77) we obtain that S is birationally equivalent with the surface

$$\mathcal{E} : y^2 = x^3 - 4f(t)x.$$

The mapping from $\mathcal{S}$ to $\mathcal{E}$ is given by

$$(u,\ v,\ w) = \Big(\frac{y}{2x},\ \frac{2x^3 + y^2}{4x^2},\ t\Big),$$

while the inverse mapping is of the form

$$(x,\ y,\ t) = (-2(u^2 - v),\ -4u(u^2 - v),\ w).$$

Applying now Theorem 2.2 we obtain the statement of our corollary. $\square$

An interesting question arises here concerning the existence of a non-trivial rational point on the surface $\mathcal{E}_f$ in case when $f(t) = at^4 + bt^2 + d$ for some $a,\ b,\ d \in \mathbb{Z}$. It is worth noting that if the equation $f(t) = 0$ has a rational root t_0, then on the surface $\mathcal{E}_f$ we have a rational curve $(x,\ y,\ t) = (u^2,\ u^3,\ t_0)$ and we can use the second part of Theorem 2.1 to construct other rational curves on $\mathcal{E}_f$. Without any difficulty we can give other infinite families of polynomials fulfilling the conditions of the second part of Theorem 2.1. For instance, if $f(t) = at^4 + bt^2 + u(v^2 - u)$, then on the curve $E_0 : y^2 = x^3 + u(v^2 - u)x$ there is a point $(u,\ uv)$ which is not of finite order if $uv \neq 0$.

One can check with computer that if $\max\{|a|,\ |b|,\ |d|\} \leq 100$, then there exists $t \in \mathbb{Q}$ such that infinitely many rational points lie on the curve $E_t : y^2 = x^3 + f(t)x$. This leads us to the following

Conjecture 2.5. *Let $a,\ b,\ d \in \mathbb{Z}$ and $f(t) = at^4 + bt^2 + d$. Then there exists $t_0 \in \mathbb{Q}$ such that there are infinitely many rational points on the curve E_{t_0}.*

3. Rational points on $\mathcal{E}^g : y^2 = x^3 + g(t)$

Let $g \in \mathbb{Q}[t]$ be a monic polynomial of degree 6 and let $g(t) \neq t^6$. For such g let us consider the surface

$$\mathcal{E}^g : y^2 = x^3 + g(t).$$

For a given $t \in \mathbb{Q}$ let us denote the curve $y^2 = x^3 + g(t)$ by E^t. Let us recall how a torsion part of the curve E^t looks like with a fixed $t \in \mathbb{Q}$ ([8], page 323). If $g(t) = 1$, then $\operatorname{Tors} E^t \cong \mathbb{Z}/6\mathbb{Z}$. If $g(t) \neq 1$ and $g(t)$ is a square in $\mathbb{Q}$, then

$\operatorname{Tors} E^t = \{\mathcal{O},\ (0,\ \sqrt{g(t)}),\ (0,\ -\sqrt{g(t)})\}$. In case when $g(t) = -432$ we have $\operatorname{Tors} E^t = \{\mathcal{O},\ (12,\ 36),\ (12,\ -36)\}$. If $g(t) \neq 1$ and $g(t)$ is a cube in $\mathbb{Q}$, then $\operatorname{Tors} E^t = \{\mathcal{O},\ (-\sqrt[3]{g(t)},\ 0)\}$. In the remaining cases we have $\operatorname{Tors} E^t = \{\mathcal{O}\}$. As an immediate consequence we obtain that if there is a rational base change $t \mapsto \beta(t)$ such that on the curve $\mathcal{E}^{g\circ\beta}$ we have the section $\sigma = (x,\ y)$ with $xy \neq 0$, then σ is a non-torsion section.

We show the following

Theorem 3.1. *Let $g \in \mathbb{Q}[t]$ be a monic polynomial of degree six. If g is not an even polynomial, then there exists a rational base change $t = \chi(u)$ such that there is a non-torsion section on the curve $\mathcal{E}^{g\circ\chi}$.*

Proof. Without loss of generality we can assume that $g(t) = t^6+at^4+bt^3+ct^2+dt+e$ for certain $a, b, c, d, e \in \mathbb{Z}$ with $b \neq 0$ or $d \neq 0$. Let now C_2 denote a curve defined over the field $\mathbb{Q}(t)$ obtained from $\mathcal{E}^g$ after substituting $x = \frac{u^2-a}{3} - t^2$. We consider the curve of the form

$$\begin{aligned} C_2 : y^2 = u^2t^4 + bt^3 - \frac{a^2-3c-2au^2+u^4}{3}t^2 + dt \\ + \frac{-a^3+27e+3a^2u^2-3au^4+u^6}{27} =: h_2(t). \end{aligned}$$

Note that the point at infinity on the curve C_2 is rational. Let us put $t = T,\ y = uT^2 + pT + q$. Then

$$(uT^2+pT+q)^2 - h_2(T) = a_0 + a_1T + a_2T^2 + a_3T^3,$$

where

$$\begin{aligned} a_0 &= \frac{a^3-27e+27q^2-3a^2u^2+3au^4-u^6}{27}, \quad a_1 = -d+2pq, \\ a_2 &= \frac{a^2-3c+3p^2+6qu-2au^2+u^4}{3}, \quad a_3 = -b+2pu. \end{aligned}$$

Solving the system of equations $a_2 = a_3 = 0$ with respect to $p,\ q$ we obtain

$$p = \frac{b}{2u},\ q = \frac{-3b^2-4a^2u^2+12cu^2+8au^4-4u^6}{24u^3}. \tag{3.1}$$

Now, if $p,\ q$ are given by (3.1), then the equation $(uT^2+pT+q)^2 - h_2(T) = 0$ has a root $T = -\chi_1(u)/\chi_2(u) =: \chi(u)$, where

$$\begin{aligned} \chi_1(u) = -27b^4 - 72b^2(a^2-3c)u^2 - 48(a^4-3ab^2-6a^2c+9c^2)u^4 \\ + 8(16a^3-9b^2-72ac+216e)u^6 - 96(a^2-3c)u^8 + 16u^{12}, \end{aligned}$$

$$\chi_2(u) = 72u^2(3b^3+4b(a^2-3c)u^2-8(ab-3d)u^4+4bu^6).$$

Our computations imply that on the curve $\mathcal{E}^{g\circ\chi}$ we have the section $\sigma_2 = ((u^2-a-3T^2)/3,\ uT^2+pT+q)$, where $p,\ q$ are given by (3.1) and $T = \chi(u)$. It is easy to see that the section σ_2 is non-torsion.

It is also worth noting that the assumption $b \neq 0$ or $d \neq 0$ is essential for the employed method because in the opposite case the function χ_2 is identically equal to zero. $\square$

Here a natural question arises whether the assumption that for a certain $t_0 \in \mathbb{Q}$ infinitely many rational points lie on the curve E^{t_0} enables to construct a rational

curve on the surface $\mathcal{E}^g$. Unfortunately, we are not able to show such a theorem with any even polynomial g. However, we can prove the following

Theorem 3.2. *Let $g \in \mathbb{Q}[t]$ be a monic and even polynomial of degree six. If there exists $t_0 \in \mathbb{Q}$ such that there are infinitely many rational points on the curve E^{t_0}, then the set of $t \in \mathbb{Q}$ such that E^t has positive rank is infinite.*

Proof. Because g is even we can assume that $g(t) = t^6 + at^4 + ct^2 + e$ for certain $a,\ c,\ e \in \mathbb{Z}$ with $a \neq 0$ or $c \neq 0$. The case $a = c = 0$ will be discussed in the next section. For the proof it will be more convenient to work with the surface $\mathcal{F}^g$ given by the equation

$$\mathcal{F}^g : Y^2 + 2t^3Y = X^3 + at^4 + ct^2 + e.$$

Let us denote $G(X,\ Y,\ t) := Y^2+2t^3Y-(X^3+at^4+ct^2+e)$. Then $\mathcal{E}^g$ is birationally equivalent with $\mathcal{F}^g$ by the mapping $(x,\ y,\ t) = (X,\ Y + t^3,\ t)$ with the inverse $(X,\ Y,\ t) = (x,\ y - t^3,\ t)$. From the assumption there exists $t_0 \in \mathbb{Q}$ such that there are infinitely many rational points on E^{t_0}. Thus, there is a rational point $(x_0,\ y_0,\ t_0)$ on the surface $\mathcal{E}^g$ such that $x_0y_0 \neq 0$. Then the point $(x_0,\ y_0 - t_0^3,\ t_0)$ is on the surface $\mathcal{F}^g$. Let us put $X = pT + x_0,\ Y = qT + y_0 - t_0^3,\ t = T + t_0$. Then

$$G(X,\ Y,\ t) = a_1T + a_2T^2 + a_3T^3 + a_4T^4,$$

where

$$\begin{aligned}
a_1 &= -3px_0^2 + 2qy_0 - 2ct_0 - 4at_0^3 - 6t_0^5 + 6t_0^2y_0,\\
a_2 &= q^2 + 6qt_0^2 - 3p^2x_0 - c + 6at_0^2 - 6t_0^4 + 6t_0y_0,\\
a_3 &= -p^3 + 6qt_0 - 4at_0 - 2t_0^3 + 2y_0,\\
a_4 &= 2q - a.
\end{aligned}$$

Solving the system of equations $a_1 = a_4 = 0$ with respect to $p,\ q$ we obtain

$$p = -\frac{2ct_0 + 4at_0^3 + 6t_0^5 - ay_0 - 6t_0^2y_0}{3x_0^2}, \quad q = \frac{a}{2}. \tag{3.2}$$

For $p,\ q$ defined in this way, the equation $G(pT + x_0,\ qT + y_0 - t_0^3,\ T + t_0) = 0$ has the root $T = 0$ and the root

$$T = -\frac{q^2 + 6qt_0^2 - 3p^2x_0 - c + 6at_0^2 - 6t_0^4 + 6t_0y_0}{-p^3 + 6qt_0 - 4at_0 - 2t_0^3 + 2y_0}. \tag{3.3}$$

From the above computations we can see that the point $(pT+x_0,\ qT+y_0-t_0^3,\ T+t_0)$ for $p,\ q$ given by (3.2) and T defined by (3.3), lies on the surface $\mathcal{F}^g$. Hence we obtain the point $P = (pT + x_0,\ qT + y_0 - t_0^3 + (T + t_0)^3,\ T + t_0)$ on the surface $\mathcal{E}^g$. Because the set of rational points on E^{t_0} is infinite, we can assume that the coordinates of the point P are nonzero, $g(T) \neq 0, -432$ and $g(T + t_0)/g(t_0)$ is not a sixth power. If we now put $t_1 = T + t_0$, then the curve E^{t_1} has infinitely many rational points.

Let us now suppose that we have already constructed $t_1,\ \ldots,\ t_n$ such that the curve E^{t_i} has a positive rank for $i = 1,\ \ldots,\ n$. Then we can apply the above procedure to the point $(x_n,\ y_n,\ t_n)$, where $(x_n,\ y_n)$ is a rational point on E^{t_n} such that T given by (3.3) fulfills the conditions: $g(T+t_n) \neq 0,\ -432$ and $g(T+t_n)/g(t_i)$ is not a sixth power for $i = 1,\ \ldots,\ n$. Why can we find such T given by (3.3) and fulfilling these conditions? Let us note that if the polynomial g does not have the root of multiplicity 5, then there are only finitely many rational points on every

curve $g(u) = g(t_i)v^6$ (of genus > 1) for $i = 1, \ldots, n$. It is an immediate consequence of the Faltings theorem ([2]). The case when g has the root of multiplicity 5 (it is a rational root then) can be easily excluded, as then the surface $\mathcal{E}^g$ is rational over $\mathbb{Q}$. Because there are infinitely many rational points on the curve E^{t_n}, we can see that T given by (3.3) can be selected to fulfill all the necessary conditions. Using now the previous reasoning we can construct an infinite set of values $t \in \mathbb{Q}$ such that E^t has a positive rank. □

Remark 3.3. Let us note that if $g(t_0) = 0$ for a rational number t_0, then the set of rational points on the curve $E^{t_0}:\ y^2 = x^3$ is parametrized by $x = u^2,\ y = u^3$. Using the reasoning from Theorem 3.2 we can easily deduce that in this case it is possible to construct a rational curve on the surface $\mathcal{E}^g$.

From the above remark we obtain the following

Corollary 3.4. *Let $h \in \mathbb{Q}[t]$ with $\deg h = 5,\ h(0) = 1$ and let us consider the surface $\mathcal{S}:\ y^2 = x^3 + h(t)$. Then, there is a rational base change $t = \gamma(u)$ such that we have a non-torsion section on the surface $\mathcal{S}^\gamma:\ y^2 = x^3 + h(\gamma(u))$.*

Proof. Let us note that the surface $\mathcal{S}$ is birationally equivalent with the surface $\mathcal{E}^g$, where $g(t) = t^6h(1/t)$. The mapping from $\mathcal{S}$ to $\mathcal{E}^g$ is given by $(x,\ y,\ t) \mapsto (x/t^2,\ y/t^3,\ 1/t)$. Because $g(0) = 0$, we can use the Remark 3.3 to obtain the statement of our corollary. □

Example 3.5. Let $g(t) = t^6 + t^2 + 1$ and let us consider the surface $\mathcal{E}^g : y^2 = x^3 + g(t)$. For $t_0 = 1$ on the curve $E^1 : y^2 = x^3 + 3$ we have a non-torsion point $P = (1,\ 2)$. Now we calculate the quantities $p,\ q$ given by (3.2) and T given by (3.3) from the proof of Theorem 3.2. We obtain $p = 16/13,\ q = -1/13,\ T = -358/169$ and next $t_1 = T + t_0 = -189/169$. Thus, we can see that on the curve

$$E^{t_1}:\ y^2 = x^3 + \frac{47 * 2085456070589}{13^{12}}$$

we have a non-torsion point

$$P = \Big(-\frac{3531}{2197},\ \frac{1137934}{4826809}\Big).$$

Similarly to the case of the surface $\mathcal{E}_f$ considered in Section 2, we can ask whether for a given polynomial g of the form $g(t) = t^6 + at^4 + ct^2 + e$ there is $t_0 \in \mathbb{Q}$ such that the curve E^{t_0} has infinitely many rational points.

In the following section we will prove that the answer to this question is positive for polynomials of the form $g(t) = t^6 + e$. Using computer we checked that if $\max\{|a|,\ |c|,\ |e|\} \leq 10$, then there exists $t \in \mathbb{Q}$ such that infinitely many rational points lie on the curve $E^t : y^2 = x^3 + t^6 + at^4 + ct^2 + e$. This leads us to the following

Conjecture 3.6. *Let $a,\ c,\ e \in \mathbb{Z}$ and $g(t) = t^6 + at^4 + ct^2 + e$. Then there exists $t_0 \in \mathbb{Q}$ such that there are infinitely many rational points on the curve E^{t_0}.*

In view of Theorem 3.2 a natural question arises

Question 3.7. *Let $g(t) = t^6 + at^4 + ct^2 + e$ and let us consider the surface $\mathcal{E}^g$. What are the conditions guaranteeing the existence of a rational base change $t = \kappa(u)$ such that there is a non-torsion section on the surface $\mathcal{E}^{g\circ\kappa}$?*

4. Some results on the diophantine equation $x^2 - y^3 - g(z) = t$

Let $g(z) = z^6 + az^4 + bz^3 + cz^2 + dz + e \in \mathbb{Z}[z]$ and let t be a variable. In this section we will deal with the diophantine equation of the form

$$x^2 - y^3 - g(z) = t. \tag{4.1}$$

We will show that if there are infinitely many rational points on the curve C : $v^2 = s^4 - 12as^2 + 48bs + 6(a^2 - 12c)$, then the equation (4.1) has infinitely many solutions in the ring of polynomials $\mathbb{Q}[t]$. In case when $g(t) = t^6 + e$, we will use this result to prove the promised theorem concerning the existence of a rational base change $t = \chi_1(s)$ such that there exists a non-torsion section on the surface $\mathcal{E}^{g\circ\chi_1}$. We will also present some results concerning the representability of integers in the form $x^2 - y^3 - g(z)$.

We start with the following

Theorem 4.1. *If there are infinitely many rational points on the curve $C : v^2 = s^4 - 12as^2 + 48bs + 6(a^2 - 12c)$, in particular if $b \neq 0$ and $a \equiv 1 \pmod 2$, then the equation* (4.1) *has infinitely many solutions in the ring of polynomials $\mathbb{Q}[t]$.*

Proof. Let us denote $G(x, y, z) := x^2 - y^3 - g(z)$ and observe that the question about solvability of the equation $G(x, y, z) = t$ in polynomials with rational coefficients is equivalent to the question about the construction of polynomials $x,\ y,\ z \in \mathbb{Q}[t]$ such that $\deg G(x(t), y(t), z(t)) = 1$. Let us now put $x = 3T^3 + pT^2 + qT + r,\ y = 2T^2 + sT + u,\ z = T$. Then

$$G(3T^3 + pT^2 + qT + r,\ 2T^2 + sT + u,\ T) = a_0 + a_1T + a_2T^2 + a_3T^3 + a_4T^4 + a_5T^5,$$

where

$$\begin{aligned}
&a_0 = r^2 - u^3 - e, \quad a_1 = -d + 2qr - 3su^2,\\
&a_2 = -c + q^2 + 2pr - 3s^2u - 6u^2, \quad a_3 = -b + 2pq + 6r - s^3 - 12su,\\
&a_4 = -a + p^2 + 6q - 6s^2 - 12u, \quad a_5 = 6(p - 2s).
\end{aligned}$$

Solving the system of equations $a_3 = a_4 = a_5 = 0$ with respect to $p,\ q,\ r$ we obtain

$$p = 2s, \quad q = \frac{a + 2s^2 + 12u}{6}, \quad r = \frac{3b - 2as - s^3 + 12su}{18}. \tag{4.2}$$

After substituting $p,\ q,\ r$ into the equation $a_2 = 0$ and solving this equation with respect to u, we obtain

$$u = \frac{3s^2 + 2a \pm \sqrt{s^4 - 12as^2 + 48bs + 6(a^2 - 12c)}}{12}. \tag{4.3}$$

Thus, we can see that if infinitely many rational points lie on the curve

$$C: \quad v^2 = s^4 - 12as^2 + 48bs + 6(a^2 - 12c) =: U(s),$$

then all but finitely many points on C, by (4.2) and (4.3), give us a triple of polynomials $x,\ y,\ z \in \mathbb{Q}[T]$ such that $G(x(T), y(T), z(T)) = a_1T + a_0$ and $a_1 \neq 0$. After substitution $T = (t - a_0)/a_1$ we obtain a solution of the equation $x^2 - y^3 - g(z) = t$. Let us also note that we always have infinitely many rational points on C when the polynomial U has multiple roots, which is equivalent to the condition $D := 25a^6 - 144a^3b^2 - 2592b^4 - 180a^4c + 5184ab^2c - 1296a^2c^2 - 1728c^3 = 0$.

Since in the case when $D = 0$, the curve C is rational over $\mathbb{Q}$, we can assume that $D \neq 0$. To show that if $b \neq 0$ and $a \equiv 1 \pmod 2$, then infinitely many rational

points lie on the curve C, we transform C into an elliptic curve with Weierstrass equation. We can do this because the point at infinity on the curve C is rational. Using the method described in [6] one more time, we birationally transform the curve C into the curve

$$E:\ Y^2 = X^3 - 72(a^2-4c)X + 64(a^3+36b^2-36ac).$$

The mapping transforming C into E is in the form

$$(s,\ v) = \Big(\frac{48b-Y}{16a-2X},\ 2a+\frac{X}{2}-\Big(\frac{48b-Y}{16a-2X}\Big)^2\Big),$$

while the inverse mapping is given by

$$(X,\ Y) = (2(-2a+s^2+v),\ 4(12b-6as+s^3+sv)).$$

Let us note that we have a rational point $P=(8a,\ 48b)$ on the curve E. Using the chord and tangent method of adding points on elliptic curve we obtain $2P = (x_1,\ y_1)$ where

$$x_1 = \frac{25a^4-256ab^2+120a^2c+144c}{16b^2},$$
$$y_1 = 48b+\frac{(5a^2+12c)(25a^4-384ab^2+120a^2c+144c^2)}{64b^3}.$$

Because $a\equiv 1 \pmod 2$ the numerator x_1 is odd, and this means, that $x_1 \in \mathbb{Q}\setminus\mathbb{Z}$. From the Nagell-Lutz theorem ([8], page 77) we know that the torsion points on the elliptic curve $y^2=x^3+px+q,\ p,\ q\in\mathbb{Z}$ have integer coordinates, therefore, we see that the point $2P$ is not of finite order. It proves that the curve E has a positive rank; and we obtain that there are infinitely many rational points on the curve C. □

Remark 4.2. After noticing that the point $P=(8a,\ 48b)$ lies on the curve E from the proof of the above theorem we suspected that this point is not of finite order for $ab\neq 0$ and any $c\in\mathbb{Z}$. As Professor Schinzel suggested, it is not true. Indeed, if $a=6p^2,\ c=p(4b-15p^3)$, then the curve E is elliptic, if $\Delta=-764411904b^2(3b-16p^3)\neq 0$. In this case the point $P=(6p^2,\ 48b)$ is a point of order three on the curve E. If we now put $p=1,\ b=1$, then $a=6,\ c=-11$. For $a,\ b,\ c$ defined in this way, the curve E is birationally equivalent with the curve $E':\ y^2 = x^3-360x+2628$. With the assistance of APECS program ([1]) we found that the rank of the curve E' is zero. Despite this, there exists a non-trivial solution of the equation $x^2-y^3-g(z)=t$ and it turns out that it is valid if $b\neq 0$; this is equivalent to the fact that the point P is not of order two. Why is it so? If $b\neq 0$, then the order of the point P equals at least 3 and s-coordinate of preimage of the point $2P$ (which is different from the point at infinity $\mathcal{O}$) equals $(5a^2+12c)/18b$. Because the expression a_1 from the proof of Theorem 4.1 depends linearly on d and is not identically equal to zero, then there is at least one $d\in\mathbb{Z}$, for which $a_1=0$ and our method does not give a solution of the equation $x^2-y^3-g(z)=t$.

It should be noted, however, that there exists a polynomial $g\in\mathbb{Z}[t]$ for which our method does not give a solution of the equation $x^2-y^3-g(z)=t$. For example, if $g(t)=t^6+6t^4+6t^3+9t^2-150t$, then the curve C is birationally equivalent with the elliptic curve $E': y^2+y=x^3-7$. We have that $\operatorname{Tors} E' = \{\mathcal{O},\ (3,4),\ (3,-5)\}$

and using APECS program once again, we find that E' has rank equal to zero. In this case, our method leads us to the identity

$$(3T^3 + 12T^2 + 33T + 25)^2 - (2T^2 + 6T + 10)^3 - g(T) = -375.$$

Now we will note several interesting corollaries of Theorem 4.1

Corollary 4.3. *If infinitely many rational points lie on the curve $C:\ v^2 = s^4 - 12as^2 + 48bs + 6(a^2 - 12c)$, then every polynomial $h \in \mathbb{Q}[t]$ can be represented in infinitely many ways in the form $x^2 - y^3 - g(z)$, where $x,\ y,\ z \in \mathbb{Q}[t]$.*

In the following corollary we give the promised proof of the theorem concerning the existence of rational curves on the surface $y^2 = x^3 + t^6 + e$.

Corollary 4.4. *Let $\mathcal{E}^g:\ y^2 = x^3 + g(t)$, where $g(t) = t^6 + e$, then there exists a rational base change $t = \chi_1(s)$ such that there is a non-torsion section on the surface $\mathcal{E}^{g\circ\chi_1}:\ y^2 = x^3 + g(\chi_1(s))$*

Proof. Let us note that if $a = b = c = 0$, then the curve C is rational and the system of equations $a_2 = a_3 = a_4 = a_5 = 0$ from the proof of Theorem 4.1 has exactly two solutions given by

$$p_1 = 2s, \quad q_1 = \frac{2s^2}{3}, \quad r_1 = \frac{s^3}{18}, \quad u_1 = \frac{s^2}{6},$$
$$p_2 = 2s, \quad q_2 = s^2, \quad r_2 = \frac{s^3}{6}, \quad u_2 = \frac{s^2}{3}.$$

For such $p_i,\ q_i,\ r_i,\ u_i,\ (i = 1,\ 2)$ we obtain the following identities

$$\Big(3T^3 + 2sT^2 + \frac{2s^2}{3}T + \frac{s^3}{18}\Big)^2 - \Big(2T^2 + sT + \frac{s^2}{6}\Big)^3 - (T^6 + dT + e) = -\frac{648e + s^6}{648} - \frac{648d + 6s^5}{648}T, \tag{4.4}$$

$$\Big(3T^3 + 2sT^2 + 2s^2T + \frac{s^3}{6}\Big)^2 - \Big(2T^2 + sT + \frac{s^2}{3}\Big)^3 - (T^6 + dT + e) = -\frac{108e + s^6}{108} - dT. \tag{4.5}$$

If now $d = 0$, then putting $T = \chi_1(s) = -(648e + s^6)/(6s^5)$ the right side of the identity (4.4) disappears and on the surface $\mathcal{E}^{g\circ\chi_1}:\ y^2 = x^3 + g(\chi_1(s))$ we obtain a section

$$\sigma = \Big(\frac{419904e^2 - 648es^6 + s^{12}}{18s^{10}},\ -\frac{272097792e^3 - 419904e^2s^6 + 1944es^{12} + s^{18}}{72s^{15}}\Big).$$

It is easy to see that the order of σ is not finite. $\square$

Let us remind that $a_1 = -d + 2qr - 3su^2$, where q, r, s, u are given by (4.2) and (4.3) from the proof of the Theorem 4.1.

Corollary 4.5. *If $d \in \mathbb{Z}$ and on the curve $C:\ v^2 = s^4 - 12as^2 + 48bs + 6(a^2 - 12c)$, there is a rational point such that $a_1 \neq 0$, then for every integer n the diophantine equation $x^2 - y^3 - g(z) = n$ has solutions in rationals $x,\ y,\ z$ such that there exists an integer L_g dependent only on the polynomial g such that $L_gx,\ L_gy,\ L_gz \in \mathbb{Z}$. In particular, for $g(z) = z^6$ we have $L_g = 24416 = 2^9 * 3^5$*

Proof. In the light of Theorem 4.1 the first part of the statement is obvious. Now putting $d = e = 0,\ s = 6$ and next $T = (n+72)/72$ into the identity (4.4) we obtain

$$\Big(\frac{n^3 - 72n^2 + 15552n + 373248}{24416}\Big)^2 - \Big(\frac{n^2 - 72n + 5184}{2592}\Big)^3 - \Big(\frac{n+72}{72}\Big)^6 = n.$$

This proves the second part of our corollary. □

Corollary 4.6. *Let $g(z) = z^6 + dz$. If $d = 1$, then for every integer n the diophantine equation $x^2 - y^3 - g(z) = n$ has infinitely many solutions in integers. If $d = -72t^5 + 1$ for a certain integer t, then for every integer n the diophantine equation $x^2 - y^3 - g(z) = n$ has a solution in integers.*

Proof. Let n be a fixed integer. If $d = -1$, then for the proof we will use the identity (4.5). Let us put $e = 0,\ s = 6t$ and $T = -432t^6 - n$. For polynomials given by

$$\begin{aligned} x(t) &= 3n^3 + 12t(-1 + 324t^5)n^2 + 36t^2(1 - 288t^5 + 46656t^{10})n \\ &\quad + 36t^3(-1 + 432t^5 - 62208t^{10} + 6718464t^{15}), \\ y(t) &= 2n^2 + 6t(-1 + 288t^5)n + 12t^2(1 - 216t^5 + 31104t^{10}), \\ z(t) &= -n - 432t^6, \end{aligned}$$

we get $x(t)^2 - y(t)^3 - g(z(t)) = n$.

If now $d = -72t^5 - 1$, then we put $e = 0,\ s = 6t,\ T = -n - 72t^6$ into the identity (4.4). We obtain that $x^2 - y^3 - g(z) = n$ for

$$\begin{aligned} x &= 3n^3 + 12t(-1 + 54t^5)n^2 + 24t^2(1 - 72^5 + 1944t^{10})n \\ &\quad + 12t^3(-1 + 144t^5 - 5184t^{10} + 93312t^{15}), \\ y &= 2n^2 + 6t(-1 + 48t^5)n + 6t^2(1 - 72t^5 + 1728t^{10}), \\ z &= -n - 72t^6. \end{aligned}$$

□

5. Rational points on some non-isotrivial elliptic surfaces

In view of our consideration it is natural to ask whether it is possible to obtain similar results for non-isotrivial elliptic surfaces of the form

$$\mathcal{E}:\ y^2 = x^3 + A(t)x + B(t),$$

where $A,\ B \in \mathbb{Q}[t] \setminus \{0\}$. If $t \mapsto \alpha(t)$ is a rational base change, then by $\mathcal{E}_\alpha$ let us denote the surface $\mathcal{E}_\alpha :\ y^2 = x^3 + A(\alpha(t))x + B(\alpha(t))$. Let us also remind that if $\mathcal{C} : y^2 = x^3 + m(t)x + n(t)$, where $m,\ n \in \mathbb{Z}[t]$ is an elliptic curve over $\mathbb{Q}(t)$, then the points of finite order on $\mathcal{C}$ have coordinates in $\mathbb{Z}[t]$.

In the following section we will prove the generalization of the first part of Theorem 2.1 and Theorem 2.2.

Theorem 5.1. *Let $\mathcal{E} : y^2 = x^3 + f_4(t)x + g_4(t)$, where $f_4,\ g_4 \in \mathbb{Q}[t]$. If $\deg f_4 = 3$ and $\deg g_4 \leq 4$ or $\deg f_4 = 4,\ \deg g_4 \leq 4$ and at least one of polynomials $f_4,\ g_4$ is not even, then there exists rational base change $t = \psi(s)$ such that we have a non-torsion section on the surface $\mathcal{E}_\psi$.*

Proof. Let us denote $H(x,\ y,\ t) := y^2 - (x^3 + f_4(t)x + g_4(t))$. Let us first consider the case when $\deg f_4 = 3$ and $\deg g_4 \leq 4$. Without loss of generality we can assume that $f_4(t) = at^3 + bt + c,\ g_4(t) = dt^4 + et^3 + ft^2 + gt + h$ for some $a,\ b,\ \ldots,\ h \in \mathbb{Z}$

with $a\neq 0$ and $g_4(t)\neq 0$. Let us put $x=pT+q,\ y=rT^2+sT+u,\ t=T$. For $x,\ y,\ t$ defined in this way we obtain

$$H(x,\ y,\ t)=a_0+a_1T+a_2T^2+a_3T^3+a_4T^4,$$

where

$$a_0=-h-cq-q^3+u^2,\quad a_1=-g-cp-bq-3pq^2+2su,$$
$$a_2=-f-bp-3p^2q+s^2+2ru,\ a_3=-e-p^3-aq+2rs,\ a_4=-d-ap+r^2.$$

Let us note that the system of equations $a_2=a_3=a_4=0$ has a solution given by

$$p=\frac{-d+r^2}{a},\quad q=-\frac{-d^3+a^3e+3d^2r^2-3dr^4+r^6-2a^3rs}{a^4},\tag{5.1}$$
$$u=-\frac{-f-bp-3p^2q+s^2}{2r}.$$

If now $p,\ q,\ u$ are given by (5.1), then the equation $H(pT+q,\ rT^2+sT+u,\ T)=0$ has a solution

$$T=-\frac{h+cq+q^3-u^2}{g+cp+bq+3pq^2-2su}=:\psi(r,s).\tag{5.2}$$

In this case we obtain a two-parametric solution of the equation defining the surface $\mathcal{E}$. For convenience let us put $r=1$ and $\psi(s):=\psi(1,s)$. Therefore, we can see that if $p,\ q,\ u$ are given by (5.1) and $T=t=\psi(s)$, then on the surface $\mathcal{E}_\psi$ we have a section $\sigma=(pT+q,\ T^2+sT+u)$. Performing affine change of variables we transform the surface $\mathcal{E}_\psi$ into the $\mathcal{E}'_\psi : y^2=x^3+f'_4(s)x+g'_4(s)$, where $f'_4,\ g'_4\in\mathbb{Z}[s]$. Then σ goes to the section σ' on $\mathcal{E}'_\psi$. It turns out that x-coordinate of the section $2\sigma'$ belongs to $\mathbb{Q}(s)\setminus\mathbb{Q}[s]$. From the remark given at the beginning of this section we see that σ' is not of finite order. We will not present here the details of this proof as it requires a lot of computations which are immensely difficult to perform without computer.

Let us now consider the case when $\deg f_4=4,\ \deg g_4\leq 4$ and at least one of the polynomials $f_4,\ g_4$ is not even. We can assume that $f_4(t)=at^4+bt^2+ct+d,\ g_4(t)=et^4+ft^3+gt^2+ht+i$, where $a,\ b,\ \dots,\ i\in\mathbb{Z},\ a\neq 0$ and at least one of the numbers $c,\ f,\ h$ is non-zero. Let now C_3 denote a curve over $\mathbb{Q}(u)$ obtained from $\mathcal{E}$ after substitution $x=(u^2-e)/a$. Hence, we consider the curve of the form

$$C_3:\ y^2=u^2t^4+ft^3+\frac{-be+ag-bu^2}{a}t^2+\frac{-ce+ah-cu^2}{a}t$$
$$+\frac{-a^2de-e^3+a^3i+(a^2d+3e^2)u^2-3eu^4+u^6}{a^3}=:V(t)$$

Now putting $y=uT^2+pT+q,\ t=T$ we obtain

$$(uT^2+pT+q)^2-V(T)=a_0+a_1T+a_2T^2+a_3T^3,$$

where

$$a_0=\frac{-a^2de-e^3+a^3i+(a^2d+3e^2)u^2-3eu^4+u^6}{a^3},\ a_1=\frac{-ce+ah-2apq+cu^2}{a},$$
$$a_2=\frac{-be+ag-ap^2-2aqu+bu^2}{a},\ a_3=f-2pu.$$

The system of equations $a_2 = a_3 = 0$ has a solution given by

$$p = \frac{f}{2u}, \quad q = \frac{-af^2 - 4beu^2 + 4agu^2 + 4bu^4}{8au^3}. \tag{5.3}$$

For $p,\ q$ defined in this way the equation $(uT^2 + pT + q)^2 - V(T) = 0$ has exactly one solution

$$T = \frac{-a^2de - e^3 + a^3i + (a^2d + 3e^2)u^2 - 3eu^4 + u^6}{a^2(-ce + ah - 2apq + cu^2)} =: \psi_1(u). \tag{5.4}$$

Now, putting $t = \psi_1(u)$ we obtain the section $\sigma_1 = ((u^2 - e)/a,\ uT^2 + pT + q)$ on the surface $\mathcal{E}_{\psi_1}$, and similarly to the previous case, we show that the order of σ_1 is not finite. □

Using similar method we can prove the following

Theorem 5.2.

(1) *Let* $\mathcal{E} : y^2 = x(x^2 + f_2(t)x + f_4(t))$, *where* $f_2,\ f_4 \in \mathbb{Q}[t]$. *If* $\deg f_2 \leq 2,\ \deg f_4 \leq 3$ *or* $\deg f_2 \leq 2,\ \deg f_4 = 4$ *and at least one of the polynomials* $f_2,\ f_4$ *is not even, then there exists a rational base change* $t = \psi(u)$ *such that we have a non-torsion section on the surface* $\mathcal{E}_\psi$.

(2) *If* $\deg f_2 = 2,\ \deg f_4 = 4$ *and there is* $t_0 \in \mathbb{Q}$ *such that the curve* $\mathcal{E}_{t_0} : y^2 = x(x^2 + f_2(t_0)x + f_4(t_0))$ *has infinitely many rational points, then there exists a rational base change* $t = \psi(u)$ *such that we have a non-torsion section on the surface* $\mathcal{E}_\psi$.

(3) *Let* $\mathcal{E} : y^2 = x(x^2 + f_4(t)x + g_4(t))$, *where* $f_4,\ g_4 \in \mathbb{Q}[t]$ *and* $\deg f_4 = \deg g_4 = 4$. *If at least one of the polynomials* $f_4,\ g_4$ *is not even, then there exists a rational base change* $t = \psi(u)$ *such that we have a non-torsion section on the surface* $\mathcal{E}_\psi$.

Proof. The proof of part (1) and (2) does not bring any difficulties, and therefore, it will be omitted (the reasoning is exactly the same as in the proof of Theorem 2.1).

We will now outline the proof of part (3) of our theorem. Let $a,\ b$ be leading coefficients of the polynomials $f_4,\ g_4$ respectively. Let us put $x = b/(u^2 - a)$ and treat $\mathcal{E}$ as a curve defined over $\mathbb{Q}(u)$. Let us denote this curve by C_4. Then, the point at infinity, say P, on C_4 is rational. Because at least one of the polynomials $f_4,\ g_4$ is not even, with the use of point P we can construct a non-torsion section on $\mathcal{E}$. □

Our previously considered elliptic surfaces (excluding the surface from part (3) of our Theorem 5.2) were rational over $\mathbb{C}$. It means that they are rational over a certain finite extension of the field $\mathbb{Q}$. It is clear that the questions about such surfaces can be also asked about general elliptic surfaces.

For $t \in \mathbb{Q}$ let us denote by $\mathcal{E}_t$ the fibre of the mapping $\pi : \mathcal{E} \to \mathbb{P}$ over t. In 1992 B. Mazur proposed an interesting conjecture concerning rational points on $\mathcal{E}$.

Conjecture 5.3. (Conjecture 4 from [5])
A family of elliptic curves $\{\mathcal{E}_t\}_{t\in\mathbb{Q}}$ *fulfills one of the following conditions*

(1) *for all but finitely many* $t \in \mathbb{Q}$ *the curve* $\mathcal{E}_t$ *has the Mordell-Weil rank equal to zero,*

(2) *for a dense set of rational numbers* t, *the Mordell-Weil rank of the curve* $\mathcal{E}_t$ *is positive.*

As pointed in [5], the only known example of a family of elliptic curves fulfilling the condition (1) of the above conjecture is the constant family with the rank equal to zero, and it seems quite probable that if the family $\{\mathcal{E}_t\}_{t\in\mathbb{Q}}$ is non-split, then (1) is not valid. Examples of families of elliptic curves fulfilling condition (2) of the above conjecture can be found in [3], [7], [4].

We believe that the following conjecture can be easier to prove

Conjecture 5.4. *For a non-constant family of elliptic curves $\{\mathcal{E}_t\}_{t\in\mathbb{Q}}$ there is $t \in \mathbb{Q}$ such that the curve $\mathcal{E}_t$ has infinitely many rational points.*

As a corollary from the above conjecture we obtain an interesting

Theorem 5.5. *Let us assume that Conjecture 5.4 is true. Then for a non-constant family of elliptic curves $\{\mathcal{E}_t\}_{t\in\mathbb{Q}}$, the set of rational numbers t such that the rank of $\mathcal{E}_t$ is positive, is infinite.*

Proof. Assuming Conjecture 5.4 to be true, we find $t_1 \in \mathbb{Q}$ such that infinitely many rational points lie on the curve $\mathcal{E}_{t_1}$. Suppose that we already constructed $t_2, \ldots, t_n$ such that the curve $\mathcal{E}_{t_i}$ for $i = 1, \ldots, n$ has infiniteley many rational points. Let us further suppose that it is possible to find a polynomial $h \in \mathbb{Q}[t]$, such that for $i = 1, \ldots, n$ the equation $h(t) = t_i$ does not have a solution in rationals and the system of equations

$$(5.5)\qquad \begin{cases} A(t_1)Y_1^4 = A(h(T_1)),\ B(t_1)Y_1^6 = B(h(T_1)), \\ \qquad\qquad\vdots \\ A(t_n)Y_n^4 = A(h(T_n)),\ B(t_n)Y_n^6 = B(h(T_n)). \end{cases}$$

does not have solutions in rationals. From the assumption there exists $t \in \mathbb{Q}$ such that the curve $\mathcal{E}_{h(t)}:\ y^2 = x^3 + A(h(t))x + B(h(t))$ has a positive rank. Defining now $t_{n+1} = h(t)$ and repeating the reasoning, we obtain the statement of our theorem.

We will show now that there exists a polynomial $h \in \mathbb{Q}[t]$ fulfilling the above conditions. Let h_1 be polynomial in $\mathbb{Q}[t]$ such that the equation $h_1(t) = t_i$ for $i = 1, \ldots, n$ does not have any solutions. Clearly it is enough to show the existence of our polynomial for the first row in the system of equations (5.5). Let us, therefore, consider the system of equations $A(t_1)Y_1^4 = A(h_1(T_1)),\ B(t_1)Y_1^6 = B(h_1(T_1))$. If $A(t_1)B(t_1) = 0$, then this system has finitely many rational solutions, and we will find $h_2 \in \mathbb{Q}[t]$ such that the polynomial $h = h_1 \circ h_2$ fulfills the desired conditions. Thus, let us assume that $A(t_1)B(t_1) \neq 0$. If $(A(h_1(T_1))/A(t_1))^3 \neq (B(h_1(T_1))/B(t_1))^2$, then our system has at most $3\deg(A \circ h_1) + 2\deg(B \circ h_1)$ solutions in $\mathbb{Q}$ and there exists $h_2 \in \mathbb{Q}[t]$ such that the polynomial $h = h_1 \circ h_2$ fulfills the desired conditions. In case when $(A(h_1(T_1))/A(t_1))^3 = (B(h_1(T_1))/B(t_1))^2$, the problem is reduced to the examination of the curve $C:\ Y_1^2 = H(T_1)$, where H is a polynomial such that $A(h_1(T_1))/A(t_1) = H(T_1)^2,\ B(h_1(T_1))/B(t_1) = H(T_1)^3$. If the polynomial H was a square of another polynomial, we would obtain that the family $\{\mathcal{E}_t\}_{t\in\mathbb{Q}}$ is constant, which contradicts the assumption. Therefore, we see that the polynomial H is not a square. Now, there exists a polynomial h_2, such that the genus of the curve $C':\ Y_1^2 = H(h_2(T_1))$ is ≥ 2. From the Faltings theorem there are only finitely many rational points on C'; so after a polynomial change of variable, we obtain a polynomial fulfilling all the necessary conditions. Applying this reasoning to the second, …, n-th equation in the system (5.5) we obtain the statement of our theorem. $\square$

In view of above theorem a natural question arises

Question 5.6. *Are conditions (2) of Conjecture 5.3 and Conjecture 5.4 equivalent?*

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